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EXISTENCE OF PERIODIC SOLUTIONS OF EQUATIONS OF MOTION OF A SOLID BODY SIMILAR TO THE LAGRANGE GYROSCOPE

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The problem of existence of periodic solutions of equations of motion of a solid body with a fixed point similar to the Lagrange gyroscope is considered. The body center of mass is displaced by a small quantity relative to the axis of symmetry, and that quantity is taken as the small parameter. Cases of existence of periodic solutions that correspond to uniform rotation about the axis of symmetry in the Lagrange solution, which can be represented by series in integral or fractional powers of the small parameter, are considered separately.

1. In conventional notation the Euler — Poisson equations of motion of a solid body about a fixed point in the Lagrange problem ($A = B$ and C are the principal moments of inertia; $x_0 = y_0 = 0$ and z_0 are coordinates of the center of mass; p, q, and r are the angular velocity components; γ_1, γ_2, and γ_3 are directional cosines of the vertical in the coordinate system attached to the body; M is the mass of body, and t is time) have the particular solution

$$p = 0, \; q = 0, \; r = r^\circ, \; \gamma_1 = 0, \; \gamma_2 = 0, \; \gamma_3 = 1 \tag{1.1}$$

Let us consider the problem of existence of periodic solutions that in the case close to the Lagrange solution correspond to solution (1.1)

$$A = B, \; x_0 = \sqrt{\mu} z_0, \; y_0 = 0 \tag{1.2}$$

where μ is a small dimensionless parameter. We introduce the dimensionless quantities

$$p = \sqrt{\mu} n^{-1} p', \quad q = \sqrt{\mu} n^{-1} q', \quad r = n^{-1} r'$$

$$\gamma_1 = \sqrt{\mu}\, \gamma_1', \quad \gamma_2 = \sqrt{\mu}\, \gamma_2', \quad \gamma_3 = \gamma_3', \quad t = nt' \left(n = \sqrt{\frac{A}{Mgz_0}}\right)$$

Taking into consideration conditions (1.2) and omitting for simplicity of notation the primes, we reduce the Euler — Poisson equations, with the fixed z-axis directed vertically upwards, to the form

$$\frac{dp}{dt} = aqr + \gamma_2, \; \frac{dq}{dt} = -apr + \gamma_3 - \gamma_1, \; \frac{dr}{dt} = -\mu b^{-1} \gamma_2 \tag{1.3}$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \; \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \; \frac{d\gamma_3}{dt} = \mu (q\gamma_1 - p\gamma_2)$$

$$a = (A - C) / A, \quad b = C/A$$

The first three integrals are of the form

$$\mu p^2 + \mu q^2 + br^2 + 2\mu\gamma_1 + 2\gamma_3 = \mu p_0^2 + \mu q_0^2 + 2\mu\gamma_{10} + br_0^2 + 2\gamma_{30} \qquad (1.4)$$
$$\mu p\gamma_1 + \mu q\gamma_2 + br\gamma_3 = \mu p_0\gamma_{10} + \mu q_0\gamma_{20} + br_0\gamma_{30}$$
$$\mu\gamma_1^2 + \mu\gamma_2^2 + \gamma_3^2 = 1$$

Solution of the first and last of Eqs. (1.4) for r and γ_3 yields

$$\gamma_3 = 1 - \mu f_1, \qquad r = r_0 - \mu f_2$$

where
$$f_1 = {}^1/_2 F_1 + {}^1/_8 \mu F_1^2 + \ldots \qquad (1.5)$$
$$f_2 = \frac{1}{2br_0}(F_2 - F_{20}) + \mu\frac{1}{8br_0}\left[\frac{1}{br_0^2}(F_2 - F_{20})^2 - (F_1^2 - F_{10}^2)\right] + \ldots$$
$$F_1 = \gamma_1^2 + \gamma_2^2, \quad F_2 = p^2 + q^2 + 2\gamma_1 - (\gamma_1^2 + \gamma_2^2)$$

Initial values of F_1 and F_2 are denoted by F_{10} and F_{20}, respectively, and the dots indicate terms of higher order of smallness with respect to μ.

Eliminating in Eq. (1.3) γ_3 and r, we obtain the following s system of four equations:

$$\frac{dp}{dt} = ar_0q + \gamma_2 - \mu aqf_2, \quad \frac{dq}{dt} = 1 - ar_0p - \gamma_1 + \mu(apf_2 - f_2) \qquad (1.6)$$
$$\frac{d\gamma_1}{dt} = r_0\gamma_2 - q - \mu(\gamma_2 f_2 - qf_1), \quad \frac{d\gamma_2}{dt} = -r_0\gamma_1 + p + \mu(\gamma_1 f_2 - pf_2)$$

By the substitution of variables

$$p = P + h\Gamma_1 + c_1, \quad q = Q + h\Gamma_2 \qquad (1.7)$$
$$\gamma_1 = (1 + \beta h)\Gamma_1 + \beta P + c_2, \quad \gamma_2 = (1 + \beta h)\Gamma_2 = \beta Q$$

we reduce system (1.6) to the form

$$dP/dt = \lambda_1 Q + \mu G_1, \quad dQ/dt = -\lambda_1 P + \mu G_2 \qquad (1.8)$$
$$\frac{d\Gamma_1}{dt} = \lambda_2\Gamma_2 + \mu G_3, \quad \frac{d\Gamma_2}{dt} = -\lambda_2\Gamma_1 + \mu G_4$$
$$\lambda_{1,2} = {}^1/_2\left[(2 - b)r_0 \pm \sqrt{b^2r_0^2 - 4}\right], \quad \beta = {}^1/_2\left(br_0 + \sqrt{b^2r_0^2 - 4}\right)$$
$$h = \frac{1}{\lambda_1 - \lambda_2}, \quad c_1 = \frac{r_0}{ar_0^2 + 1}, \quad c_2 = \frac{1}{ar_0^2 + 1}$$
$$G_1 = -(1 + \beta h)aqf_2 - h(qf_1 - \gamma_2 f_2), \quad G_2 = -(1 + \beta h) \times (f_1 - apf_2) + h(pf_1 - \gamma_1 f_2)$$
$$G_3 = \beta aqf_2 + qf_1 - \gamma_2 f_2, \quad G_4 = \beta(f_1 - apf_2) - pf_1 + \gamma_1 f_2$$

The generating system (with $\mu = 0$) for Eqs. (1.8) has pure imaginary roots when $b^2r_0^2 - 4 \geqslant 0$, as assumed throughout the following analysis.

2. Let $\lambda_1 / \lambda_2 = n_1 / n_2$ be a rational number; this can be achieved by, for example, a suitable selection of r_0. The general solution for this generating system is then periodic of period $T_0 = 2\pi n_1 / \lambda_1 = 2\pi n_2 / \lambda_2$. Let us formulate the problem of determining the $T(\mu)$-periodic solutions of system (1.8) with fairly small μ which for $\mu = 0$ would reduce to a solution of period T_0 of the generating system.

We substitute variable τ for t, setting $t = (1 + \mu\alpha)\tau$, where α is a function of the small parameter μ which is to be determined. The problem now reduces to the determination of periodic solutions of period T_0 of the new system of equations [1]

$$\begin{aligned}
&dP/d\tau = \lambda_1 Q + \mu H_1, \quad dQ/dt = -\lambda_1 P + \mu H_2 \qquad (2.1)\\
&d\Gamma_1/d\tau = \lambda_2\Gamma_2 + \mu H_3, \quad d\Gamma_2/d\tau = -\lambda_2\Gamma_1 + \mu H_4\\
&H_1 = (1 + \mu\alpha) G_1 + \alpha\lambda_1 Q, \quad H_2 = (1 + \mu\alpha) G_2 - \alpha\lambda_1 P\\
&H_3 = (1 + \mu\alpha)\ G_3 + \alpha\lambda_2\Gamma_2, \quad H_4 = (1 + \mu\alpha)\ G_4 - \alpha\lambda_2\Gamma_1
\end{aligned}$$

We seek a solution of system (2.1) of the form

$$\begin{aligned}
&P(\tau) = M_1 \cos\lambda_1\tau + M_2 \sin\lambda_1\tau + \Sigma_1 \qquad (2.2)\\
&Q(\tau) = -M_1 \sin\lambda_1\tau + M_2\cos\lambda_1\tau + \Sigma_2\\
&\Gamma_1(\tau) = M_3\cos\lambda_2\tau + \Sigma_3, \quad \Gamma_2(\tau) = -M_3\sin\lambda_2\tau + \Sigma_4\\
&\left(\Sigma_i = \sum_{n=1}^{\infty} C_i^{(n)}(\tau)\mu^n, \quad i = 1, 2, 3, 4\right)\\
&P(0) = M_1 = M_1^\circ + m_1, \quad Q(0) = M_2 = M_2^\circ + m_2\\
&\Gamma_1(0) = M_3 = M_3^\circ + m_3, \quad \Gamma_2(0) = 0
\end{aligned}$$

The periodic solutions of system (1.8) which correspond to the T_0-periodic solutions of system (2.1) are of period $T = (1 + \mu\alpha)\,T_0$. We represent function α as $\alpha = \alpha_0 + m_4$. In accordance with Poincare's method we vary the initial conditions which in this case coincide with the arbitrary constants of solution of the generating system. We also vary α so as to have solution (2.2) of periodic form, and seek m_1, m_2, m_3, and m_4 in the form of functions of the small parameter μ which vanish for $\mu = 0$.

3. The coefficients $C_i^{(n)}(\tau)$ are determined by equations

$$\begin{aligned}
&\frac{dC_1^{(n)}(\tau)}{d\tau} = \lambda_1 C_2^{(n)}(\tau) + H_1^{(n)}(\tau), \quad \frac{dC_2^{(n)}(\tau)}{d\tau} = -\lambda_1 C_1^{(n)}(\tau) + H_2^{(n)}(\tau) \qquad (3.1)\\
&\frac{dC_3^{(n)}(\tau)}{d\tau} = \lambda_2 C_4^{(n)}(\tau) + H_3^{(n)}(\tau), \quad \frac{dC_4^{(n)}(\tau)}{d\tau} = -\lambda_2 C_3^{(n)}(\tau) + H_4^{(n)}(\tau)
\end{aligned}$$

with initial conditions $C_i^{(n)}(0) = 0$; $H_i^{(n)}(\tau)$ are known functions when $C_i^{(k)}(\tau)$ are determined for $k < n$.

It is possible to establish for system (3.1) the validity of the following relationships:

$$\begin{aligned}
&C_i^{(n)}(\tau) = \sum_{\alpha=1}^{4} S_\alpha^{(n)}(\tau)\,\varphi_{i\alpha}(\tau) \qquad (3.2)\\
&S_i^{(n)}(\tau) = \int_0^\tau \sum_{\alpha=1}^{4} \varphi_{\alpha i}(u)\,H_\alpha^{(n)}(u)\,du\\
&C_i^{(n)}(T_0) = S_i^{(n)}(T_0) \quad (i = 1, 2, 3, 4)
\end{aligned}$$

where $\varphi_{\alpha i}(\tau)$ are the elements of the generating system fundamental matrix, and

$\varphi_{ii}(0) = 1$ and $\varphi_{ij}(0) = 0$ $(i \neq j)$.

Substitution of the first approximations for P, Q, Γ_1, and Γ_2 into (1.7) by formulas (1.5) yields

$$F_1^{(0)} = c_2^2 + \beta^2(M_1^2 + M_2^2) + (1+\beta h)^2 M_3^2 + 2\beta c_2 \times (M_1 \cos\lambda_1\tau + M_2 \sin\lambda_1\tau) + 2c_2(1+\beta h) M_3 \cos\lambda_2\tau + 2\beta \times (1+\beta h) \times [M_1 M_3 \cos(\lambda_1 - \lambda_2)\tau + M_2 M_3 \sin(\lambda_1 - \lambda_2)\tau] \tag{3.3}$$

$$F_2^{(0)} = c_1^2 + 2c_2 + M_1^2 + M_2^2 + hM_3^2 + 2(c_1 + \beta) \times (M_1 \cos\lambda_1\tau + M_2 \sin\lambda_1\tau) + 2[c_1 h + (1+\beta h)] M_3 \cos\lambda_2\tau + 2h[M_1 M_3 \cos(\lambda_1 - \lambda_2)\tau + M_2 M_3 \sin(\lambda_1 - \lambda_2)\tau] - F_1^{(0)}$$

For brevity of presentation we introduce the quantities L_i

$$\begin{aligned} L_1 &= -(1+\beta h) af_2 - h(f_1 - \beta f_2) \\ L_1' &= -(1+\beta h) ahf_2 - h[f_1 h - (1+\beta h) f_2] \\ L_2 &= -(1+\beta h)(f_1 - ac_1 f_2) + h(c_1 f_1 - c_2 f_2) \\ L_3 &= \beta ahf_2 + hf_1 - (1+\beta h) f_2, \quad L_3' = \beta ah + f_1 - \beta f_2 \\ L_4 &= \beta(f_1 - ac_1 f_2) - c_1 f_1 + c_2 f_2 \end{aligned} \tag{3.4}$$

If in these formulas we eliminate terms that are independent of μ and determined by the generating solution, they assume the following form:

$$L_i^{(0)} = [k_{i1} + k_{i2}(M_1^2 + M_2^2) + k_{i3} M_3^2] + k_{i4}(M_1 \cos\lambda_1\tau + M_2 \sin\lambda_1\tau) + k_{i5} M_3 \cos\lambda_2\tau + k_{i6} M_3 [M_1 \cos(\lambda_1 - \lambda_2)\tau + M_2 \sin(\lambda_1 - \lambda_2)\tau]$$

Here and in what follows k_{rs} denote functions of parameters a and r_0 which can be determined by formulas (3.4), (1.5), and (3.3). They are not adduced here because of their unwieldiness.

Functions $S_i^{(1)}(\tau)$ are calculated by the second formula (3.2) as follows:

$$S_1^{(1)}(\tau) = \int_0^\tau [M_2(L_1^{(0)} + \alpha\lambda_1) - \sin\lambda_1 u L_2^{(0)} + M_3 \sin(\lambda_1 - \lambda_2) u L_1'^{(0)}] du \tag{3.5}$$

$$S_2^{(1)}(\tau) = -\int_0^\tau [M_1(L_1^{(0)} + \alpha\lambda_1) - \cos\lambda_1 u L_2^{(0)} + M_3 \cos(\lambda_1 - \lambda_2) u L_1'^{(0)}] du$$

$$S_3^{(1)}(\tau) = \int_0^\tau \{L_3^{(0)}[-M_1 \sin(\lambda_1 - \lambda_2) u + M_2 \cos(\lambda_1 - \lambda_2) u] - \sin\lambda_2 u L_4^{(0)}\} du$$

$$S_4^{(1)}(\tau) = -\int_0^\tau \{M_3(L_3'^{(0)} + \alpha\lambda_2) + [M_1 \cos(\lambda_1 - \lambda_2) u + M_2 \sin(\lambda_1 - \lambda_2) u] L_3^{(0)} - \cos\lambda_2 u L_4^{(0)}\} du$$

4. It is shown in [1] that if solution (2.2) is to be T_0 -periodic it is necessary and sufficient that

$$\Psi_1 = P(T_0) - P(0) = 0, \quad \Psi_2 = Q(T_0) - Q(0) = 0 \qquad (4.1)$$
$$\Psi_3 = \Gamma_1(T_0) - \Gamma_1(0) = 0, \quad \Psi_4 = \Gamma_2(T_0) - \Gamma_2(0) = 0$$

where Ψ_i $(i = 1, 2, 3, 4)$ are functions of M_1, M_2, M_3, α, and μ. The equalities (4.1), which determine $M_j^{(0)}$, α_0, and m_i $(j = 1, 2, 3;\ i = 1, 2, 3, 4)$, are not independent owing to the existence in system (2.1) of the first integral which corresponds to the second formula in (1.4) [2]. It can be shown that the third condition is a corollary of the remaining if $M_3 \neq 0$, as well as when $M_1 = M_2 = M_3 = 0$ for $ar_0^2 + 1 > 0$. By analogy with the statement in [3] it is possible to consider one of the quantities M_1°, M_2°, M_3°, or α as an arbitrary constant, and one of the m_i $(i = 1, 2, 3, 4)$ as an arbitrary function of μ which vanishes for $\mu = 0$.

Reducing equalities (4.1) by μ and equating to zero the terms at zero powers of μ, we obtain the following necessary conditions of periodicity:

$$C_i^{(1)}(T_0) = C_i^{(1)}(M_1, M_2, M_3, \alpha) = 0 \quad (i = 1, 2, 3)$$

which in accordance with the last of formulas (3.2) are of the form

$$M_2E_1 + R_1 = 0, \quad M_1E_1 + R_2 = 0, \quad M_3E_3 + R_4 = 0 \qquad (4.2)$$

where

$$E_1 = k_{11} - {}^1/_2\ k_{24} + k_{12}(M_1^2 + M_2^2) + (k_{13} + {}^1/_2\ k_{16}')M_3^2 + \alpha\lambda_1$$
$$E_3 = k_{31} - {}^1/_2\, k_{45} + (k_{32} + {}^1/_2\, k_{36})\ (M_1^2 + M_2^2) + k_{33}M_3^2 + \alpha\lambda_2$$

The expressions for R_1, R_2, and R_4 are nonzero only when λ_1 / λ_2 is equal 2, ${}^1/_2$, or -1, and are of the form

$$R_1 = 0, \quad R_2 = {}^1/_2\ k_{15}'M_3^2, \quad R_4 = {}^1/_2\ (k_{35} - k_{46})M_1M_2 \quad (\lambda_1 / \lambda_2 = 2)$$
$$R_1 = {}^1/_2\ (k_{26} - k_{14}')M_2M_3, \quad R_2 = -{}^1/_2\ (k_{26} - k_{14}')M_1M_3$$
$$R_4 = {}^1/_2\, k_{34}(M_1^2 - M_2^2) \quad (\lambda_1 / \lambda_2 = {}^1/_2)$$
$$R_1 = -{}^1/_2\ k_{16}'M_2M_3, \quad R_2 = M_3[k_{11}' + k_{12}'\ (M_1^2 + M_2^2) + k_{13}'M_3^2] + {}^1/_2\ k_{16}'M_3^2M_1 - {}^1/_2\, k_{25}M_3, \quad R_4 = -{}^1/_2 k_{14}M_1$$
$$(\lambda_1 / \lambda_2 = -1)$$

Let M_1°, M_2°, M_3°, and α_0 satisfy Eqs. (4.2). Let us consider Jacobi's matrices of $C_1(T_0)$, $C_2(T_0)$, and $C_4(T_0)$ in terms of M_1, M_2, M_3, and α calculated for $M_j = M_j^\circ$ $(j = 1, 2, 3)$, $\alpha = \alpha_0$, and also of Ψ_1, Ψ_2, and Ψ_4 in terms of m_i with $m_i = \mu = 0$ $(i = 1, 2, 3, 4)$. The calculation of the second matrix does not involve differentiation with respect to μ, hence it is possible to set $\mu = 0$, and since M_j $(j = 1, 2, 3)$, α, and m_i $(i = 1, 2, 3, 4)$ appear in solutions in the form of related sums, the considered matrices are the same. We denote them by J.

The solution of Eqs. (4.1) comprises the following three cases of existence of periodic solutions.

1°. $M_1^\circ = M_2^\circ = M_3^\circ = 0$, $E_1 \neq 0$, and $E_3 \neq 0$. The matrix J is then of the third rank, and there exist univalent functions m_1, m_2, m_3 of α and μ that satisfy Eqs. (4.1). When μ is fairly small these functions can be represented in the form of converging series in integral powers of μ which vanish when $\mu = 0$; α is an arbitrary constant, except $\alpha_0 = -\lambda_1^{-1}(k_{11} - {}^1/_2\, k_{24})$ or $\alpha_0 = -\lambda_2^{-1}(k_{31} - {}^1/_2\, k_{45})$. Since α_0 and m_4 appear in the solution in the form of sum $\alpha = \alpha_0 + m_4$, it is possible to set the arbitrary quantity m_4 equal zero without affecting the sought solutions. In the considered case solution (2.2) is periodic with arbitrary parameter α and is analytic with respect to μ in some neighborhood of its zero value.

2°. If $M_1^\circ = M_2^\circ = E_3 = 0$, $E_1 \neq 0$, and $M_3 k_{33} \neq 0$, matrix J is of the third rank, M_3 is an arbitrary quantity, and $\alpha_0 = \lambda_2^{-1}({}^1/_2\, k_{45} - k_{31} - k_{33}M_3^2)$. Equations (4.1) have solutions in the form of series in integral powers of μ for m_1, m_2, and m_4 that depend on the arbitrary M_3, and vanish when $\mu = 0$ (m_3 is to be taken as equal zero).

3°. If $M_3 \neq 0$ and the ratio λ_1 / λ_2 is equal 2, ${}^1/_2$, or -1, matrix J is of the third rank, unless specified otherwise, M_1 is an arbitrary constant, and m_2, m_3, and m_4 can be determined in the form of series of the required form in integral powers of μ.

If the rank of J is specified to be lower than the third, cases of branching are possible [4], and there exist solutions which can be represented for reasonably small μ by converging series in fractional powers of μ.

Let

$$\lambda_1 / \lambda_2 = 2, \quad M_1^\circ = M_2^\circ = 0, \quad M_3^\circ \neq 0, \quad E_3 = 0 \tag{4.3}$$
$$E_1 = {}^1/_2 (k_{26} - k_{14}') M_3^0 \neq 0, \quad E_1 - {}^1/_2\, (k_{26} - k_{14}')\, M_3^0 = 0$$

then the necessary conditions of periodicity (4.2) are satisfied and J is of the second rank; M_1°, M_2°, M_3°, and α are determined by conditions (4.3) and m_1, m_2, m_3, and m_4 remain to be determined. Applying the theorem on implicit functions to the first and fourth of Eqs. (4.1), we obtain for m_2 and m_4 a unique solution of the form

$$m_2 = \sum_{i+j\geqslant 1} \beta_{ij} m_1^i \mu^j, \quad m_4 = \sum_{i+j\geqslant 1} \alpha_{ij} m_1^i \mu^j$$

We set $m_3 = \delta\mu$, where δ is an arbitrary constant. Substituting the expressions for m_2, m_3, and m_4 into the second of Eqs. (4.1) we obtain equations of branching of the form

$$\left[k_{12} - 2\left(k_{32} + \frac{1}{2} k_{36} + \frac{k_{34}}{2M_3^\circ}\right)\right] m_1^3 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} R_{ij} m_1^i \mu^j = 0$$

which has one small real solution for m_1 which can be presented in the form of series in powers of $\mu^{1/3}$ and depends on the arbitrary parameter δ.

If we now set $m_3 = \delta_1 m_1$, the equation of branching assumes the form

$$\left[2\left(k_{13} + \frac{1}{2} k_{16}'\right) M_3^\circ - 4k_{33}M_3^\circ - \frac{1}{2}(k_{26} - k_{14}')\right] \delta_1 m_1^2 +$$

$$\left[k_{12}+\left(k_{13}+\frac{1}{2}k_{16}'\right)\delta_1^2+\alpha_{20}\lambda_1\right]m_1^3+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}R_{ij}m_1^i\mu^j=0$$

$$\alpha_{20}=-\frac{1}{\lambda_2}\left(k_{32}+\frac{1}{2}k_{36}+k_{33}\delta_1^2+\frac{k_{34}}{2M_3^\circ}\right)$$

If the coefficient at m_1^2 is nonzero, then for sign $R_{01}R_{02}=-1$, i.e. for

$$\operatorname{sign} C_2^{(2)}(M_1^\circ,\ M_2^\circ,\ M_3^\circ,\ \alpha_0)\cdot C_2^{(3)}(M_1^\circ,\ M_2^\circ,\ M_3^\circ,\ \alpha_0)=-1$$

there are two small real solutions for m_1 which can be represented in the form of power series in $\mu^{1/2}$ [4]. These solutions contain the arbitrary parameter δ_1. If we set $m_3=\delta_1 m_1+\delta_2 m_1^2$ and select δ_1 so that the coefficient at m_1^2 in the equation of branching vanishes, there is one real solution which can be represented in the form of power series in $\mu^{1/3}$, is determinate in some neighborhood of zero, and depends on the arbitrary parameter δ_2.

We represent m_3 in the form of the sum

$$m_3=\sum_k \delta_k m_1^k$$

By selecting δ_k so that the coefficients in the equation of branching successively vanish it is possible to obtain within the range of initial conditions of Eqs. (2.1) a sequence of branching points of periodic solutions which can be represented in the form of series in fractional powers of μ. If such sequence converges, we obtain the concentration point of periodic solution branching. By imposing other constraints it is possible to obtain other cases of branching.

5. If the ratio λ_1/λ_2 is neither 2, $^1/_2$, or -1, the solution of the problem of existence of periodic solutions requires the consideration of higher approximations than in the two cases considered above. For this we use formulas (3.2) which yield

$C_1^{(2)}(T_0)$ and $C_2^{(2)}(T_0)$

$$C_1^{(2)}(T_0)=M_2\alpha(E_1-\lambda_1)T_0+\int_0^{T_0}[M_2L_1^{(1)}-\sin\lambda_1\tau L_2^{(1)}+$$
$$M_3\sin(\lambda_1-\lambda_2)\tau L_1^{(1)}+L_1^{(0)}S_2^{(1)}(\tau)+L_1^{'(0)}S_4^{(1)}(\tau)+\lambda_1\alpha S_2^{(1)}(\tau)]\,d\tau$$

$$C_2^{(2)}(T_0)=-M_1\alpha(E_1-\lambda_1)T_0-\int_0^{T_0}[M_1L_1^{(1)}-\cos\lambda_1\tau L_2^{(1)}+$$
$$M_3\cos(\lambda_1-\lambda_2)\tau L_1^{'(1)}+L_1^{'(0)}S_3^{(1)}(\tau)+\lambda\alpha S_1^{(1)}(\tau)+L_1^{(0)}S_1^{(1)}(\tau)]\,d\tau$$

Instead of the second periodicity condition of (4.1) we consider the equality

$$\Psi_2^*=\frac{1}{M_2}\Psi_1+\frac{1}{M_1}\Psi_2=0 \qquad (5.1)$$

Terms with zero powers of μ do not appear in (5.1) and the coefficients at first powers of μ are of the form

$$\frac{1}{M_2}C_1^{(2)}(T_0)+\frac{1}{M_1}C_2^{(2)}(T_0)=T_0\left\{\left(\frac{\eta_4}{M_2}-\frac{\eta_3}{M_1}\right)[k_{11}'+\right.$$
$$k_{12}'(M_1^2+M_2^2)+k_{13}'M_3^2]-\frac{k_{15}'M_1M_3}{2\lambda_2}-$$

$$\frac{k_{14}'M_3}{4\lambda_1 M_1}(k_{46}+k_{35})(M_1^2+M_2^2)-\frac{k_{15}'M_3}{2\lambda_2 M_1}[k_{41}+k_{42}(M_1^2+M_2^2)+$$
$$k_{43}M_3^2-\frac{k_{34}}{2}(M_1^2-M_2^2)]+\frac{k_{16}'M_3}{(\lambda_1-\lambda_2)M_1}[k_{31}'+$$
$$k_{32}'(M_1^2+M_2^2)+k_{33}'M_3^2(M_1^2+M_2^2)+\frac{k_{44}}{2}(M_1^2-M_2^2)]-$$
$$\left.\frac{k_{15}^2M_3^2}{2\lambda_2}\left(\frac{M_1}{M_2}-\frac{M_2}{M_1}\right)-\frac{k_{16}k_{25}M_3^2}{2(\lambda_1-\lambda_2)}+\frac{k_{15}'k_{46}M_3^2}{2(\lambda_1-2\lambda_2)}\right\}$$

where η_3 and η_4 represent the lower bound values of integrals in the second and third of equalities (3.5).

After reduction of equality (5.1) by μ we obtain the following necessary conditions for the existence of solutions of the sought form:

$$\frac{1}{M_2}C_1^{(2)}(T_0)+\frac{1}{M_1}C_2^{(2)}(T_0)=0 \qquad (5.2)$$

Condition (5.2) together with the first and third of conditions (4.1) are also sufficient conditions of periodicity, since the rank of Jacobi's matrix in zero of Ψ_1, Ψ_2^*, and Ψ_3 is equal three with respect to $m_i\ (i=1, 2, 3, 4)$. One of the quantities M_1, M_2, M_3, or α can then be arbitrarily selected, and m_i which corresponds to remaining quantities can be represented in the form of series in integral powers of μ which are convergent when μ is fairly small, satisfy Eqs.(4.1) and (5.1), and vanish when $\mu=0$. If the rank of J is required to be lower than the third, cases of branching are possible.

6. Let us now assume that the frequency ratio λ_1/λ_2 is an irrational number. The generating system for Eqs. (2.1) has the particular solution

$$P(\tau)=0,\ Q(\tau)=0,\ \Gamma_1(\tau)=M_3\cos\lambda_2\tau,\ \Gamma_2(\tau)=-M_3\sin\lambda_2\tau \qquad (6.1)$$

with frequency λ_2 .

The conditions of existence of periodic solutions of system (2.1) which reduce for $\mu=0$ to solution (6.1) are of the form

$$\Psi_1=m_1(\cos\lambda_1T_0-1)+m_2\sin\lambda_1T_0+\mu C_1^{(1)}(T_0)+\ldots=0$$
$$\Psi_2=-m_1\sin\lambda_1T_0+m_2(\cos\lambda_1T_0+1)+\mu C_2^{(1)}(T_0)+\ldots=0$$
$$\Psi_4=C_4^{(1)}(T_0)+\ldots=0$$

Solutions of the derived equations are of the same form as in case 2°. A similar statement is also valid for the other periodic solution of the generating system with frequency λ_1.

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